Investigation of some inequality.

https://www.linkedin.com/feed/update/urn:li:activity:6783649531583107072

Prove that if $a, b, c > 0$ such that $abc = 1$ and $n \in \mathbb{N}$, $\lambda \geq 1$ then

$$\frac{1}{a^n + \lambda} + \frac{1}{b^n + \lambda} + \frac{1}{c^n + \lambda} \leq \frac{3}{1+\lambda}.$$

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First note that original problem need some correction related to λ.

Indeed, inequality $\sum \frac{1}{a^n + \lambda} \leq \frac{3}{1+\lambda}$ isn't holds for $\lambda = 1, n = 1$

because for $a = b = \frac{1}{5}, c = 25$ we have

$$\sum \frac{1}{a+1} = \frac{2}{\frac{1}{5}+1} + \frac{1}{25+1} = \frac{133}{78} > \frac{3}{2}.$$

As it will be seen from further the value of natural n isn't matter but we should claim $\lambda \geq 2$.

Since $\sum \frac{1}{a^n + \lambda} \leq \frac{3}{1+\lambda} \Leftrightarrow \sum \frac{1}{\frac{a^n}{\lambda} + 1} \leq \frac{3}{1 + \frac{1}{\lambda}}$ then denoting $t := \frac{1}{\lambda}$

and $x := \frac{a^n}{\lambda} = a^n t$ we obtain that

$$\begin{cases} \sum \frac{1}{a^n + \lambda} \leq \frac{3}{1+\lambda} \\ abc = 1, a, b, c > 0 \end{cases} \Leftrightarrow \begin{cases} \sum \frac{1}{x+1} \leq \frac{3}{1+t} \\ xyz = t^3, x, y, z > 0 \end{cases}.$$

1.Necessary condition for t :

Let $x = y = \frac{1}{n}$ and $z = t^3 n^2, n \in \mathbb{N}$ then we have

$$\frac{2n}{1+n} + \frac{1}{1+t^3n^2} \leq \frac{3}{1+t} \Rightarrow \lim_{n\to\infty}\left(\frac{2n}{1+n} + \frac{1}{1+t^3n^2}\right) = 2 \leq \frac{3}{1+t} \Leftrightarrow t \leq \frac{1}{2}.$$

2. Now we will prove that for any positive $t \leq \frac{1}{2}$ holds inequality

(**1**) $\quad \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \leq \frac{3}{1+t}$, where $x, y, z > 0$ and $xyz = t^3$.

But first we will prove that $\frac{1}{1+x} + \frac{1}{1+y} \leq \frac{2}{1+\sqrt{xy}}$, $x, y > 0$ and $xy \leq 1$.

Indeed, $\frac{1}{1+x} + \frac{1}{1+y} \leq \frac{2}{1+\sqrt{xy}} \Leftrightarrow \frac{2+x+y}{1+x+y+xy} \leq \frac{2}{1+\sqrt{xy}} \Leftrightarrow$

$2 + x + y + 2\sqrt{xy} + \sqrt{xy}(x+y) \leq 2 + 2x + 2y + 2xy \Leftrightarrow$

$2\sqrt{xy}\left(1 - \sqrt{xy}\right) \leq (x+y)\left(1 - \sqrt{xy}\right) \Leftrightarrow$

$\left(1 - \sqrt{xy}\right)\left(x + y - 2\sqrt{xy}\right) \geq 0.$

Coming back to (**1**) we assume that $xy = \min\{xy, yz, zx\}$.

Then $x^3y^3 \leq xy \cdot yz \cdot zx = t^6 \Leftrightarrow xy \leq t^2$ and denoting $a := \sqrt{xy}$,

we obtain $0 < a \leq t$, $z = \frac{t^3}{a^2}$ and inequality $\frac{1}{1+x} + \frac{1}{1+y} \leq \frac{2}{1+a}$

which holds because $a < 1$. Hence we have

$$\frac{3}{1+t} - \left(\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}\right) \geq \frac{3}{1+t} - \frac{2}{1+a} - \frac{a^2}{a^2+t^3} =$$

$\frac{(t-a)^2(a(2-t) + t(1-2t))}{(1+a)(1+t)(a^2+t^3)} \geq 0$ ($0 < t \leq \frac{1}{2}$ implies $a(2-t) + t(1-2t) \geq 0$).

Since inequality $\sum \frac{1}{x+1} \leq \frac{3}{1+t}$ holds for any $x,y,z > 0$ such that $xyz = t^3$ iff $0 < t \leq \frac{1}{2} \Leftrightarrow \lambda \geq 2$ then corrected version of the problem must be:

Prove that if $a,b,c > 0$ such that $abc = 1$ and $n \in \mathbb{N}$, $\lambda \geq 2$ then

$$\frac{1}{a^n+\lambda} + \frac{1}{b^n+\lambda} + \frac{1}{c^n+\lambda} \leq \frac{3}{1+\lambda}.$$